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Creation of a mirror space

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1 Introduction

The decay of the false vacuum under the effects of gravitation was studied first by the authors of [1]. In the semi-classical limit, the decay probability is given by

$$P = Ae^{-S_E(\phi_b, g_b)+S_E(\phi_0, g_0)}, \tag{1.1}$$

where (ϕ_b, g_b) is configuration of the scalar field for the bounce solution and the background metric, and (ϕ_0, g_0) is configuration for the false vacuum solution. S_E is the Euclidean action.

The idea that our universe may be created from nothing, rather than from the false vacuum, was proposed by [2], and it was developed to the 'theory'[3] that the universe as a space-like cross section of a configuration of (ϕ, g) is described by the wave function

$$\Psi = \sum e^{-S_E(\phi, g)}, \tag{1.2}$$

where the summation is over all possible configuration of (ϕ, g) that is specified on the given space-like surface and have no boundary at past. In the semi-classical approximation, the contribution to the summation comes dominantly from the classical solution and if the configuration continues analytically to a bounce solution, the probability of the universe is given by

$$P = Ae^{-S_E(\phi_b, g_b)}. \tag{1.3}$$

In this paper, I study the probability of the creation of *a mirror space* that is a compact curved space with a domain wall. It is shown that such a mirror space is created naturally.

We review in section 2 and 3, the junction condition and the equation of motion for the domain wall [4][5][6][7]. In section 4, we define the mirror space and find the bounce solution for the mirror space. In section 5, we calculate the Euclidean action for the bounce solution and the probability of the creation for the mirror space through this bounce solution. Section 6 is devoted to discuss our result. I use the unit $8\pi G = 1$.

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2 Junction equation

A point on the surface of a spherically symmetric domain wall is described by $x^i = (\tau, \theta, \phi)$ with the line element,

$$ds^2 = -d\tau^2 + r(\tau)^2 d\Omega^2, \tag{2.1}$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$. Introducing η, a coordinate orthogonal to the surface of the domain wall, a point in the neighborhood of the wall is described by $x^\mu = (\eta, x^i)$ with the line element,

$$ds^2 = d\eta^2 - d\tau^2 + r(\tau)^2 d\Omega^2. \tag{2.2}$$

In this coordinate system, the Einstein equations can be divided to 3+1 form as

$$G_{\eta\eta} = T_{\eta\eta}, \quad G_{i\eta} = T_{i\eta}, \quad G_{ij} = T_{ij}. \tag{2.3}$$

We use the Latin index i, j (and k, m later) to indicate the components orthogonal to η. In the thin wall approximation, the stress-energy tensor is written as

$$T_{\mu\nu} = S_{\mu\nu}\delta(\eta) + \text{reguler term}, \tag{2.4}$$

where $S_{\mu\nu}$ is described in terms of the surface energy σ and the tension ζ of the domain wall as

$$S_{\mu\nu} = \sigma(\tau)u_\mu u_\nu - \zeta(\tau)(h_{\mu\nu} + u_\mu u_\nu).$$

Here u^μ is the 4-velocity of the domain wall, and $h^{\mu\nu}$ is the metric projected on the domain wall,

$$h^{\mu\nu} = g^{\mu\nu} - \xi^\mu \xi^\mu. \tag{2.5}$$

$\xi^\mu(x)$ is a unit vector orthogonal to the η =const hyper surface.

$T_{\mu\nu}$ near the domain wall is formed by the potential term proportinal to $g_{\mu\nu}$ and the kinetic term proportinal to $\partial_\eta\phi\partial_\eta\phi$, so that T_{ij} is proportional to h_{ij} and we get

$$\sigma(\tau) = \zeta(\tau). \tag{2.6}$$

In company with this equation, the conservation of the energy momentum leads that σ is τ independent, so that the energy momentum tensor is written as

$$T_{ij} = -\sigma h_{ij}\delta(\eta) + \text{reguler term}. \tag{2.7}$$

We can recast the Einstein tensor to 3+1 form also. In the Gauss-Codazzi formalism, G_{ij} turns to

$$G_{ij} = {}^{(3)}G_{ij} - \partial_\eta(K_{ij} - h_{ij}K) - [KK_{ij} - \frac{1}{2}h_{ij}(K_{km}K^{km} + K^2)], \tag{2.8}$$

where the extrinsic curvature is defined by

$$K_{ij} = -\Gamma^{\eta}_{ij} = \frac{1}{2}\partial_\eta g_{ij}, \tag{2.9}$$

and

$$K = g^{ij}K_{ij}. \tag{2.10}$$

Substituting (2.7) and (2.8) to the last equation of (2.3), the singular part reduces to

$$\partial_\eta(K_{ij} - h_{ij}K) = \sigma h_{ij}\delta(\eta), \tag{2.11}$$

or

$$\partial_\eta K_{ij} = -\frac{1}{2}\sigma h_{ij}\delta(\eta). \tag{2.12}$$

This is the junction condition, which gives the jump of the extrinsic curvatures of the outer spacetime ($\eta > 0$) and of the inner spacetime ($\eta < 0$) separated by the domain wall.

3 Motion of a domain wall

Consider a domain wall which metric is given by (2.1) and the outer and inner spacetime are O(3) symmetric. These line elements are described by

$$ds^2_\pm = f^{-1}_\pm(r)dr^2 - f_\pm(r)dt^2_\pm + r^2 d\Omega^2, \tag{3.1}$$

where $\pm$ stands for the outer and inner spacetime respectively. In each coordinate system, a point on the domain wall is expressed by $x^\mu_\pm = (r, t_\pm, \theta, \phi)$ and its velocity is

$$u^\mu_\pm = (\dot{r}, \dot{t}_\pm, 0, 0). \tag{3.2}$$

where the over dot denotes a derivative with respect to the proper time of the domain wall.

In this coordinate system, the extrinsic curvature (2.9) reduces to

$$K^\pm_{ij} = \frac{1}{2}\xi^\mu_\pm \partial_\mu g_{ij} = \frac{1}{2}\xi^r_\pm \partial_r g_{ij}. \tag{3.3}$$

Explicit forms of $\theta\theta$ and $\phi\phi$ components are

$$K^\pm_{22} = \xi_\pm r, \tag{3.4}$$

$$K^\pm_{33} = \xi_\pm r \sin\theta, \tag{3.5}$$

and the junction condition (2.12) is written as

$$\xi_+ - \xi_- = -\frac{1}{2}\sigma r, \tag{3.6}$$

where we briefed $\xi^r_\pm$ as $\xi_\pm$. From the orthogonal conditions $g_{\mu\nu}u^\mu_\pm\xi^\nu_\pm = 0$ and the normal conditions of $u^\mu_\pm$ and $\xi^\mu_\pm$, we obtain

$$\xi^2_\pm = f_\pm + \dot{r}^2, \tag{3.7}$$

and (3.6) reduces to

$$\dot{r}^2 + f_- - (\sigma r)^{-2}[f_+ - f_- - (\sigma r/2)^2]^2 = 0. \tag{3.8}$$

This is the equation of motion for the domain wall in the Lorenzian time.

It is easy to continue it to the Euclidean equation. By Wick rotation, the line element of the inner and outer spacetime becomes

$$ds^2_\pm = f^{-1}_\pm(r)dr^2 + f_\pm(r)dt^2_\pm + r^2 d\Omega^2, \tag{3.9}$$

where $t_\pm$ is the Euclidean time. Then (3.8) reduces to

$$-\dot{r}^2 + f_- - (\sigma r)^{-2}[f_+ - f_- - (\sigma r/2)^2]^2 = 0, \tag{3.10}$$

where the dot denotes a derivative for the Euclidean proper time.

4 Bounce solution for a mirror space

Assume a Z(2) symmetric potential $V(|\phi|)$ for a real field ϕ with two degenerate vacua at $\phi = \pm\phi_0$. The domain wall solution gives a configuration to connect these two vacua.

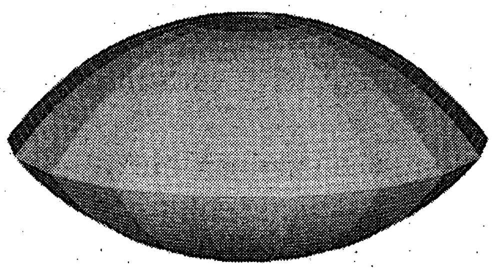

Figure 1: The de Sitter mirror space embedded in a flat space, where the coordinate ϕ is omitted and θ is taken to cover from 0 to 2π. The domain wall locates at the edge and glues the face and the back.

A compact space can be constructed by connecting a pair of compact O(3) symmetric 3-spaces, a face space ($\phi = \phi_0$) and a back space ($\phi = -\phi_0$)

that are enclosed by the domain wall. See Figure 1. In such a construction, the face and the back are duplicate each other except the sign of ϕ and have the same cosmological costant. We have called this compact space the *mirror space.*

To study the creation of the mirror space, we will find a bounce solution that mediates from nothing to the mirror space, and subsequently from the mirror space to nothing. See Figure 2(b).

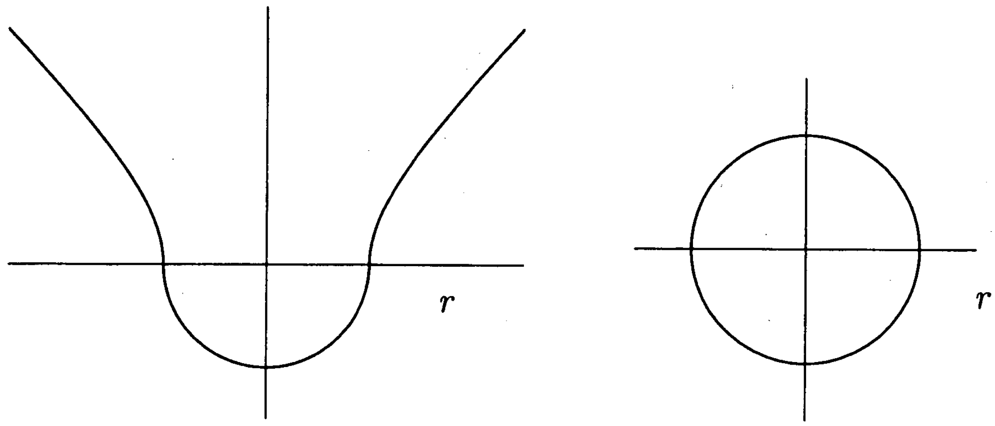

Figure 2: (a) The trajectory of the domain wall in the Lorentzian time (the upper half) and in the Euclidean time (the lower half). (b) The bounce solution in the Euclidean time.

Since $f_+ = f_-$ and $\xi_+ = -\xi_-$ for the mirror space and (3.8) does not change by the over all sign of $\xi_\pm$, omitting the suffix of $\pm$ in $\xi_\pm$ and $f_\pm$ we can write the line elements of the face and the back as

$$ds^2 = f^{-1}(r)dr^2 + f(r)dt^2 + r^2 d\Omega^2, \tag{4.1}$$

and (3.8) as

$$-\dot{r}^2 + f - a_0^{-2} r^2 = 0, \tag{4.2}$$

where

$$a_0 = (\sigma/4)^{-1}. \tag{4.3}$$

The O(3) symmetric homogeneous curved space is described by

$$f = 1 - \frac{\Lambda}{3} r^2, \tag{4.4}$$

where it gives the de Sitter space when $\Lambda > 0$ and the anti-de Sitter space when $\Lambda < 0$.

The equation of motion for the domain wall (4.2) reduces to

$$-\dot{r}^2 + 1 - a^{-2}r^2 = 0. \tag{4.5}$$

The solution is

$$r = a\sin(a^{-1}\tau), \tag{4.6}$$

where

$$a = (a_0^{-2} + \Lambda/3)^{-1/2}. \tag{4.7}$$

For the de Sitter space $a < a_0$, and for the anti-de Sitter space $a > a_0$. In both cases, the Euclidean spacetime enclosed by this trajectory of the domain wall is clearly compact and we can recognize it as a bounce solution*.

The radial size a of the bounce solution is the size of the mirror space at the creation. It is clear that the size of the anti-de Sitter mirror space is larger than the size of the de Sitter mirror space.

The line element of this bounce solution is

$$ds^2 = a^2(d\tau^2 + \sin^2\tau d\Omega^2). \tag{4.8}$$

We can recast this bounce solution in terms of the coordinates of (4.1). From the definition of the proper time, we get

$$\dot{t}^2 = f^{-2}(f - \dot{r}^2), \tag{4.9}$$

which gives a relation of r and t as follows

$$\frac{\dot{t}}{\dot{r}} = (1 - a^{-2}r^2)^{-1/2} f^{-1} a_0^{-1} r, \tag{4.10}$$

where the sign ambiguity of the square root is absorbed into the definition of t.

Integrating (4.10), we get a solution for the flat space as

$$a_0^{-2}t^2 = 1 - a_0^{-2}r^2. \tag{4.11}$$

For the curved case, using $\chi = \sqrt{3/|\Lambda|}$, we describe a solution for the de Sitter space

$$a_0^{-2}\chi^2 \tan^2(\chi^{-1}t) = 1 - a^{-2}r^2, \tag{4.12}$$

*When $\Lambda/3 < -a_0^2$, (4.5) reduces to $-\dot{r}^2 + 1 + a^{-2}r^2 = 0$, which has no bounce solution. The bounce solution is formed by the shrinking force of the domain wall in the Euclidean time. The negative cosmological constant weakens this shrinking force, while the positive cosmological constant strengthens. For $\Lambda/3 < -a_0^2$, the expanding force of the negative cosmological constant surpasses the shrinking force so that the bounce solution can not be formed.

and

$$a_0^{-2}\chi^2\tanh^2(\chi^{-1}t) = 1 - a^{-2}r^2, \tag{4.13}$$

for the anti-deSitter space.

The radial size a given by (4.7) and the time size given by the above solution decrease monotonically as Λ increases from $-3a_0^{-2}$ to ∞.

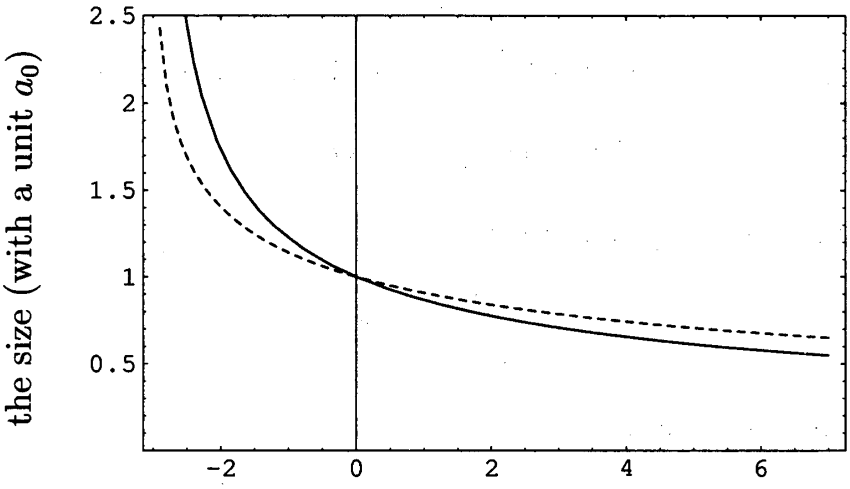

Figure 3: The solid line shows the radial size and the dot line shows the time size of the bounce solution as a function of Λ.

5 Probability of the creation of mirror spaces

The probability of the creation of the space from nothing is given by

$$P = Ae^{-S_E(\phi_b, g_b)}. \tag{5.1}$$

The Euclidean action is calculated by

$$S_{\mathrm{E}} = \int \sqrt{g}dx^4 \left[-\frac{1}{2}R - L_{\mathrm{m}}\right], \tag{5.2}$$

where the boundary surface term is omitted because the mirror space is compact.

The relevant equations are the Lagrangian of the scalar field ϕ;

$$L_{\mathrm{m}} = -\frac{1}{2}g^{\mu\nu}\phi_{,\mu}\phi_{,\nu} - V(\phi), \tag{5.3}$$

the Einstein equation;

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = T_{\mu\nu}, \tag{5.4}$$

where

$$T_{\mu\nu} = \phi_{,\mu}\phi_{,\nu} - g_{\mu\nu}\left(\frac{1}{2}\phi^{,\lambda}\phi_{,\lambda} + V(\phi)\right), \tag{5.5}$$

and the field equation;

$$(-g)^{-1/2}\left[(-g)^{1/2}\phi^{,\mu}\right]_{,\mu} - \frac{dV(\phi)}{d\phi} = 0. \tag{5.6}$$

These equations are described using the coordinate system of

$$ds^2 = d\eta^2 + a^2 d^{(3)}\Omega^2, \tag{5.7}$$

near the domain wall. Eq.(5.6) reduces to

$$\phi'' - \frac{dV(\phi)}{d\phi} = 0, \tag{5.8}$$

and integrating it, we obtain

$$\frac{1}{2}\phi'^2 - V(\phi) = V_0, \tag{5.9}$$

where V_0 is the potential at the vacua $\phi = \pm\phi_0$.
Taking the trace of the equation (5.4), we get

$$R = \phi'^2 + 4V(\phi), \tag{5.10}$$

where the dash denotes $d/d\eta$, and the Euclidean action becomes a simple form,

$$S_{\rm E} = -\int \sqrt{g}V(\phi)dx^4. \tag{5.11}$$

Here S_i^j is calculated from (5.5) as

$$\sigma = -\int_{-\epsilon}^{\epsilon} T_\theta^\theta d\eta = \int_{-\epsilon}^{\epsilon} \left(2V(\phi) + V_0\right) d\eta = 2\int_{-\epsilon}^{\epsilon} V(\phi)d\eta. \tag{5.12}$$

The contribution from the domain wall is given by

$$S_{\rm E} = -\frac{1}{2}\sigma \int \sqrt{h}dx^3. \tag{5.13}$$

Thus we get

$$S_{\rm E} = -\frac{1}{2}\sigma \int_{\rm S} \sqrt{h}dx^3 - 2\Lambda \int_{\rm V} \sqrt{g}dx^4, \tag{5.14}$$

where we used the relation $V_0 = \Lambda$. S denotes the 3-surface of the bounce solution and V denotes the spacetime of the face (or the back) edged by S.

The factor 2 of the second term comes from the sum of the face and the back.

The bounce solution of the domain wall forms a 3-sphere of a radius $r = a$, so that

$$S = 2\pi^2 a^3. \tag{5.15}$$

The volume of the face enclosed by the wall trajectory is

$$V = \frac{4\pi}{3} \int r(t)^3 dt. \tag{5.16}$$

Since $\dot{t}/\dot{r}$ given by (4.10) is a one-valued function of r, it reduces to

$$V = \frac{4\pi}{3} \int r^3 (\dot{t}/\dot{r}) dr, \tag{5.17}$$

or

$$V = \frac{\pi^2}{2} a_0^{-1} a^5 \zeta(a^2\Lambda/3). \tag{5.18}$$

Thus, the Euclidean action of the bounce solution becomes

$$S_E = -4\pi^2 a_0^{-1} a^3 - \frac{4}{3}\pi^2 \Lambda a_0^{-1} a^5 \zeta(a^2\Lambda/3). \tag{5.19}$$

Here ζ is a deformation factor of the bounce solution caused by the space curvature;

$$\zeta(x) = \frac{2}{\pi} \int_0^\pi (1 - x\sin^2\theta)^{-1} \sin^4\theta d\theta, \tag{5.20}$$

which grows monotonically as x increases.

The Euclidean action was given as a function of two parameters a_0 and Λ. Here we can consider that a_0 is first determined from the parameter of the elementary particle theory and quest the probability of the creation for the measure of Λ only. I calculated (5.19) numerically and show the result in Figure (4). It shows that most likely created is the de Sitter mirror space with $\Lambda = -3a_0^{-2}$.

Let me make sure the asymptotic behavior of (5.19). For $a \to 0$ ($\Lambda \to \infty$), $\zeta < \Lambda a_0^2$ is satisfied and

$$|\text{the second term of } S_E| < 3\pi^2 a_0 a, \tag{5.21}$$

so that S_E approaches 0 as $a \to 0$. For $a \to \infty$ ($\Lambda \to -3a_0^2$), with the help of the approximation $\zeta \approx -x^{-1} - 2x^{-2}$ at $x \to -\infty$, (5.19) becomes

$$S_E \simeq -24\pi^2 |\Lambda|^{-1} a_0^{-1} a, \tag{5.22}$$

which falls toward $-\infty$ as $a \to \infty$.

The Euclidean action for the flat case reduces to

$$S_E = -4\pi^2 a_0^{-1} a^3, \tag{5.23}$$

which is given by the authors of ref.[8].

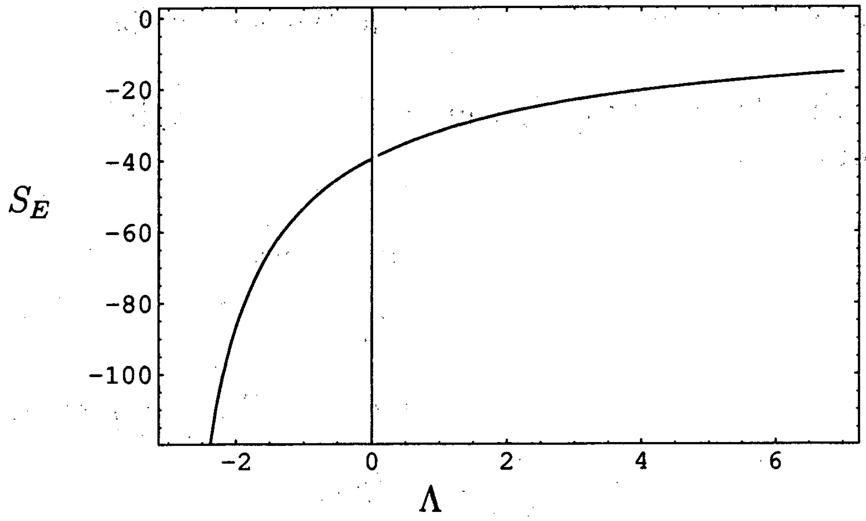

Figure 4: The Euclidean action S_E for the de Sitter mirror space ($\Lambda > 0$) and for the anti-de Sitter mirror space ($\Lambda < 0$).

6 Discussion

It was shown that the mirror space is created naturally through the bounce solution and the most likely creation is for the anti-de Sitter one with the cosmological constant $\Lambda \sim -\sigma^2$. The probability of the creation for this mirror space is infinitely large and it surpasses the creation of spaces with other geometries, for instance, the de Sitter space with no domain wall where $S_E = -24\pi^2\Lambda^{-1}$[9].

It may be a baffling result that the larger bounce solution is more likely created. We cannot consider physically the bounce solution with the infinite size. So, we must expect that its maximum is determined from some physics in future.

Is there any way to dodge this unsettlement? The above result is directly from the assumption of the no-boundary condition. Linde insisted [10] that the wave function should be given rather, by the anti-Wick rotation as

$$\Psi = \sum e^{S_E(\phi, g)}, \tag{6.1}$$

so that the probability of the creation is as follows

$$P = Ae^{S_E(\phi_b, g_b)}. \tag{6.2}$$

If we adopt this, the most likely space turns to the de Sitter mirror space with an infinitely large cosmological constant with $S_E = 0$, which is harmless because of the suppression by the other possible spaces with $S_E > 0$.

The further study on the creation of the mirror space will provide an important clue to decide which is the adequate boundary condition for the wave function of the universe.

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