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CONSIDERATION ON THE TEACHING MATERIALS OF HIGH SCHOOL MATHEMATICS. VIII

— Variations of Solutions of a Quadratic Equation for Variation of One Coefficient —

Kiyosi YAMAGUTI

Kyushu Sangyo University, Higashi-ku, Fukuoka 813, Japan

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Abstract. A solution of a quadratic equation $ax^2+bx+c=0$ is constructed in the complex plane in the sense of Hamilton by using parallelism and description of circle. When one coefficient among the coefficients $a(>0)$, b, c changes continuously, while the others are fixed, loci of the solutions of quadratic equation are considered.

1. Introduction

In the lower secondary school mathematics, a quadratic function is teached in a form $y=ax^2$ from a viewpoint of y is directly proportional to x^2. The Quadratic Formula of a quadratic equation is teached without referring an imaginary solution.

In the subject "Mathematics I" in the upper secondary school mathematics, a quadratic function is teached in a general form $y=ax^2+bx+c$, wherein to draw a graph of this function and to study a relation between the graph of general quadratic function $y=ax^2+bx+c$ and the one of canonical quadratic function $y=ax^2$ are important topics.

For a quadratic function $y=ax^2+bx+c$, change one coefficient continuously, while the others are fixed. Then, to study a deformation of the graph and a locus of the vertex of parabola are traditional problems. T.G. Edward [1] studies a teaching of a deformation of the graph for a approaching 0 and the locus of the vertex of parabola for changing b or c by using graphing calculator. In [3], we considered when the coefficient a (resp. b) changes continuously, while the others are fixed, to construct geometrically the vertex of parabola from a (resp. b), and studied a variation of the vertex for a variation of a (resp. b) by using this construction.

This paper is a continuation of our previous paper [3], and study a problem analogous to the one above for a solution of quadratic equation. The purpose of this paper is to consider geometrically variations of the solutions for a variation of one coefficient of given quadratic equation $ax^2+bx+c=0$. For this purpose, in § 2, we construct a solution of the quadratic equation in the orthogonal coordinate plane by using parallelism and description of a circle as preparation. At first, the imaginary solution points of quadratic equation are constructed, next, thanks to this construction the real solution points are constructed. In §§ 3, 4, 5, we consider the loci of solution points for a variable a, b, and c, while the others are fixed respectively.

If a solution of quadratic equation is imaginary, then we consider the orthgonal coordinate plane as a complex plane in the sense of Hamilton, hence the solution is denoted as a point (u, v), which means that the solution is real (resp. imaginary) according to $v=0$ (resp. $v\neq0$). Throughout the paper, we assume $a>0$ without loss of generality for our purpose. Notice that the graphs of $y=ax^2+bx+c$ and $y=ax^2-bx+c$ are symmetric with respect to the y-axis.

2. Geometric construction of the solutions of quadratic equation

A solution of the quadratic equation $ax^2+bx+c=0$ is obtained algebraically by a factorization of the quadratic expression ax^2+bx+c or the Quadratic Formula. The solution is also obtained geometrically as an intersection of the x-axis and a parabola $y=ax^2+bx+c$. If these have no intersection, then the solution of the equation is imaginary one. In [2], we considered geometric construction of the solution of quadratic equation.

Here, we construct the imaginary solution of the quadratic equation by using parallelism and description of a circle in two means, and by using this construction, we construct a real solution. In the complex plane in the sense of Hamilton, an imaginary point is denoted as (u, v), $v \neq 0$, but a real point is denoted as $(u, 0)$. There, it seems that a geometric construction of imaginary solution gets easier than a construction of real solution.

2. 1. Construction of imaginary solutions

For a quadratic equation

$$ax^2+bx+c=0, \tag{2.1}$$

$a>0$, assume the discriminant b^2-4ac of (2.1) is negative, then $ac>0$ and (2.1) has two non-equal imaginary solutions.

2.1.1. Construction of solutions. **A**

Case I: $b=0$

Let P be an x-intercept of a line passing through a point $(0, c)$, which is parallel to a line joining points $(0, a)$, $(1, 0)$. Then, the coordinate of P is $(c/a, 0)$. Draw a circle with points $(-1, 0)$, $P(c/a, 0)$ as the end points of diameter. Then, an intersection of this circle and the y-axis is $(0, -(c/a)^{1/2})$, $(0,(c/a)^{1/2})$. Hence, the solutions of (2.1) are obtained in the complex plane in the sense of Hamilton. (Figure 2.1)

Case II: $b \neq 0$

In the complex plane in the sense of Hamilton, under the condition

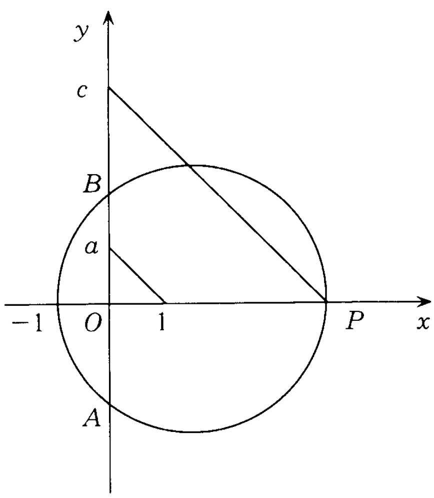

Fig. 2. 1 Solutions A, B for $b=0$

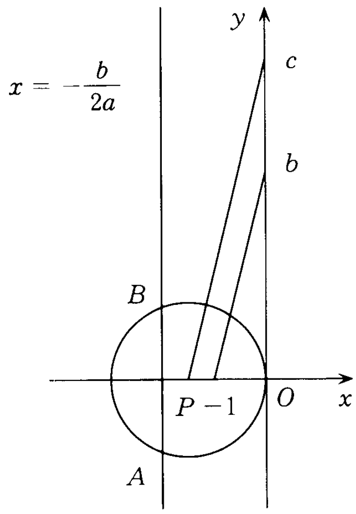

Fig. 2. 2 Solutions A, B for $b \neq 0$

$b^2-4ac<0$, a point (u, v) is a solution point of the quadratic equation (2. 1) is equivalent to that (u, v) satisfies the condition:

$$\begin{cases} x=-b/2a, \\ (x+(c/b))^2+y^2=(c/b)^2. \end{cases} \tag{2.2}$$

In fact, under the assumption $b^2-4ac<0$, the solutions of (2.1) are $(-b/2a, \pm((c/a)-(b/2a)^2)^{1/2})$, hence $u:=-b/2a, v:=\pm((c/a)-(b/2a)^2)^{1/2}=\pm(-(2cu/b)-u^2)^{1/2}$ satisfy the condition (2.2), and the converse is valid.

We construct the solution (u, v) of (2.2). Let P be an x-intercept of a line passing through point $(0, c)$, which is parallel to a line joining points $(0, b), (-1, 0)$. Then, P has a coordinate $(-c/b, 0)$. Draw a circle with P as center and radius $|c/b|$. Then, the intersections A, B of this circle and a line $x=-b/2a$ are the solution points of (2.2):

$$A(-b/2a, -((c/a)-(b/2a)^2)^{1/2}),$$
$$B(-b/2a, ((c/a)-(b/2a)^2)^{1/2}).$$

(Figure 2. 2)

2. 1. 2. Construction of solutions. **B**

In the complex plane in the sense of Hamilton, under the condition $b^2-4ac<0$, a point (u, v) is a solution point of the quadratic equation (2. 1) is equivalent to that (u, v) satisfies the condition:

$$\begin{cases} x=-b/2a, \\ x^2+y^2=c/a. \end{cases} \tag{2.3}$$

In fact, the solutions of (2.1) are (u, v), where $u:=-b/2a, v:=\pm((c/a)-(b/2a)^2)^{1/2}$, from which the result follows easily.

We construct the solution (u, v) of (2.3). Let P be an x-intercept of a line passing through point $(0, c)$, which is parallel to a line joining points $(0, a), (1, 0)$. Then, P has a coordinate $(c/a, 0)$. Draw a circle with points $(-1, 0)$, $P(c/a, 0)$ as the end points of diameter. Then, the intersections of the circle and the y-axis are $C(0, -(c/a)^{1/2})$, $D(0, (c/a)^{1/2})$. Draw a circle with points C, D as the end points of diameter, which means that the origin is the center of this circle. Then, the intersections A, B of this circle and a line $x=-b/2a$ are the solutions of (2.3), hence of equation (2.1). (Figure 2. 3)

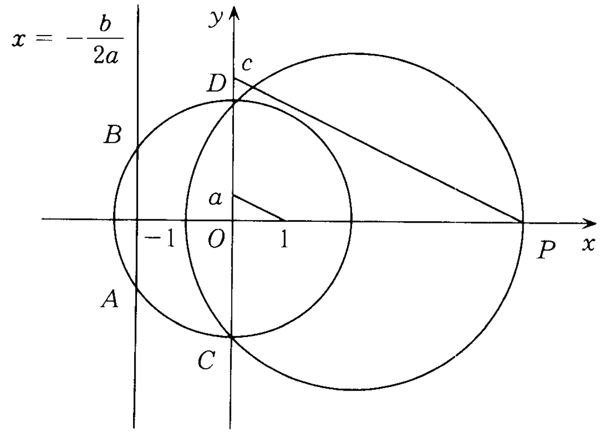

Fig. 2. 3 Solutions A, B of (2. 3)

2. 2. Construction of multiple solution

Assume $b^2-4ac=0$. Then, the quadratic equation $ax^2+bx+c=0$ has a multiple solution $-b/2a$. Two constructions of the solution are given.

2. 2. 1. Construction of solution. **A**

If $b=0$, then $c=0$ and the origin is the solution of the equation.

If $b\neq0$, then let P be an x-intercept of a line passing through the point $(0, b)$, which is parallel to a line joining points $(0, a)$, $(-\frac{1}{2}, 0)$. The abscissa of P is $-b/2a$, which is the solution of the equation. (Figure 2. 4)

2. 2. 2. Construction of solution. **B**

If $b=0$, then the origin is a solution of the equation.

If $b\neq0$, then $c\neq0$. Let P be an x-intercept of a line $y=(b/2)x+c$, then

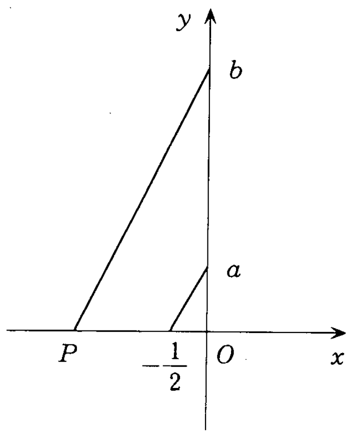

Fig. 2. 4 Construction of $P(-b/2a, 0)$. **A**

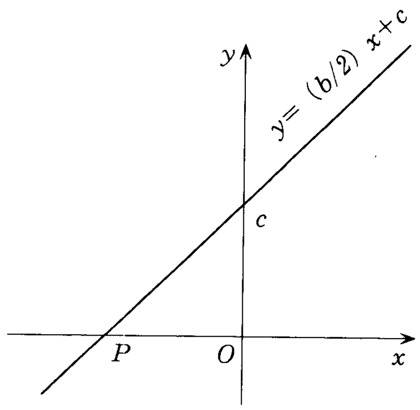

Fig. 2. 5 Construction of $P(-b/2a, 0)$. **B**

P is a solution point of the equation, since an abscissa of P is $-2c/b=-b/2a$ from the assumption $b^2=4ac$. (Figure 2. 5)

2. 3. Construction of real solutions

Assume $b^2-4ac>0$. Then, the quadratic equation $ax^2+bx+c=0\cdots$ (2.1) has two non-equal real solutions $(-(b/2a)-((b/2a)^2-(c/a))^{1/2}, 0)$, $(-(b/2a)+((b/2a)^2-(c/a))^{1/2}, 0)$. Following [2, § 4.4, p. 212] associate with (2.1) a quadratic equation

$$ax^2+bx+c^*=0, \qquad (2.1)^*$$

where $c^*:=(b^2/2a)-c$. Then, the equation $(2.1)^*$ has two non-equal imaginary solutions since the discriminant of $(2.1)^*$ is $4ac-b^2$. By using the construction of imaginary solutions in § 2.1.2, the imaginary solutions of equation $(2.1)^*$ are constructed.

Draw a circle with points $(-1, 0)$, $(c^*/a, 0)$ as the end points of diameter. The points $C(0, -(c^*/a)^{1/2})$, $D(0, (c^*/a)^{1/2})$ are the intersections of the circle and the y-axis. Draw a circle with the origin as center, radius $(c^*/a)^{1/2}$, that is, $x^2+y^2=c^*/a$. Let A^*, B^* be the intersections of this circle and a line $x=-b/2a$. Then, their coordinates are

$$A^*(-b/2a, -((c^*/a)-(b/2a)^2)^{1/2}),$$
$$B^*(-b/2a, ((c^*/a)-(b/2a)^2)^{1/2}).$$

Let A, B be the intersections of a circle with A^*, B^* as the end points of diameter and the x-axis, then

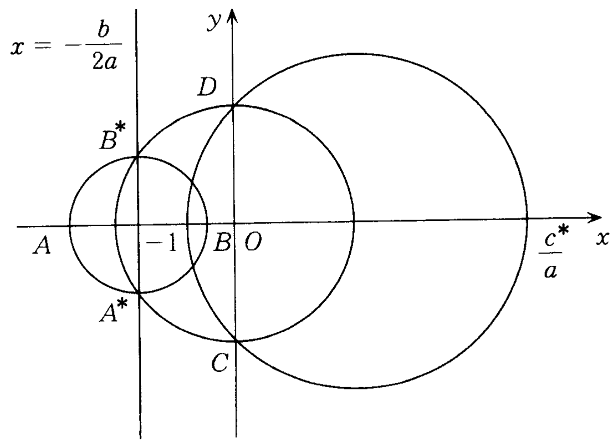

Fig. 2. 6 Solutions A, B of $ax^2+bx+c=0$

$$A(-(b/2a)-((c^*/a)-(b/2a)^2)^{1/2}, 0),$$
$$B(-(b/2a)+((c^*/a)-(b/2a)^2)^{1/2}, 0)$$

are the real solutions of quadratic equation $ax^2+bx+c=0$, since $(c^*/a)-(b/2a)^2=(b/2a)^2-(c/a)$. (Figure 2. 6) Notice that the points A, B are obtained also as x-intercept of a line which passes B^*, A^* and with slope 1 respectively.

3. Variations of solutions of $ax^2+bx+c=0$ for variation of a, while b, c are fixed

For the quadratic equation

$$ax^2+bx+c=0, \tag{2.1}$$

$a>0$, we consider the variations of solutions for a continuous variation of the coefficient a, while b, c remain fix. We consider this for two cases $b=0$ and $b\neq0$.

3. 1. Case I: $b=0$

Assume $c>0$, then the discriminant of (2. 1) is negative and (2. 1) has two non-equal imaginary solutions $A(0, -(c/a)^{1/2})$, $B(0, (c/a)^{1/2})$. Hence, we see that

if $a\to+0$, then $A\to(0, -\infty)$, $B\to(0, +\infty)$,

if $a\to+\infty$, then $A\to O$, $B\to O$,

therefore, the variation of solutions for a is explained as Figure 3. 1.

Assume $c=0$, then the solution is the origin O.

Assume $c<0$, then (2. 1) has two non-equal real solutions $A(-(-c/a)^{1/2}, 0)$, $B((-c/a)^{1/2}, 0)$. Hence, we see that

if $a\to+0$, then $A\to(-\infty, 0)$, $B\to(+\infty, 0)$,

if $a\to+\infty$, then $A\to O$, $B\to O$.

3. 2. Case II: $b\neq0$

3. 2. 1. Case of $a>0$, $b>0$, $c>0$

We consider the variation of solutions for a variation of a in the following three cases:

$a=b^2/4c$, $a>b^2/4c$, $0<a<b^2/4c$.

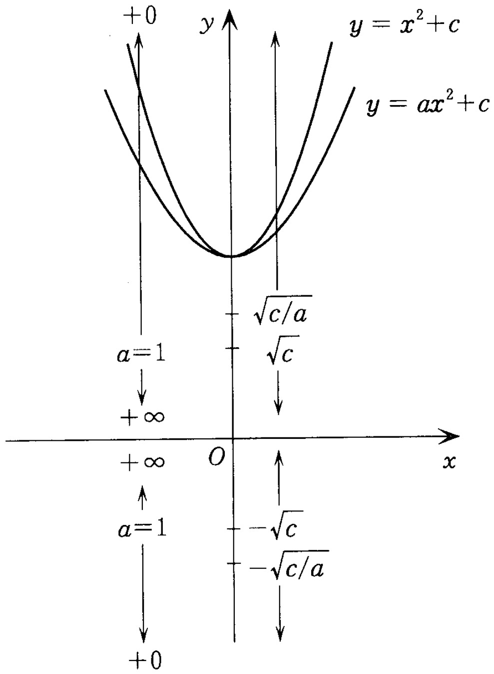

Fig. 3. 1 Variation of solutions $(0, \pm(c/a)^{1/2})$ for a, where $b=0$

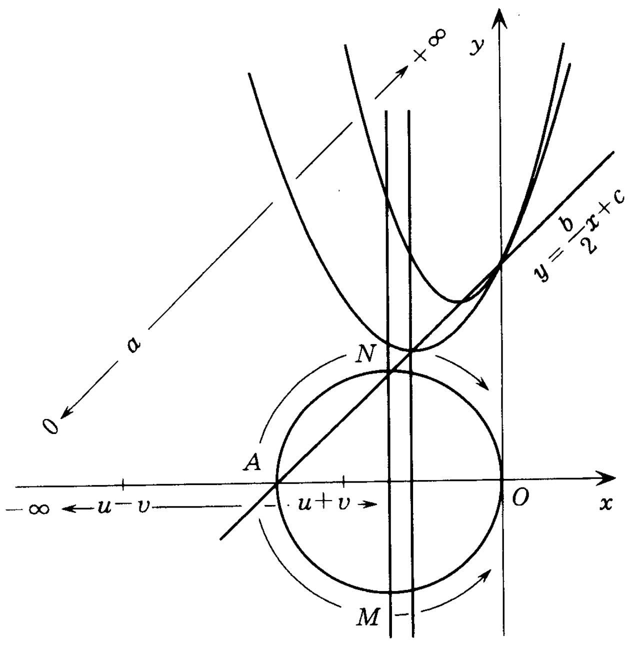

Fig. 3. 2 Variation of solutions for a, where $a>0$, $b>0$, $c>0$

If $a=b^2/4c$, quadratic equation $ax^2+bx+c=0$ has a multiple solution $A(-2c/b, 0)$.

Assume the variable a satisfies the condition $a>b^2/4c$. Then, the quadratic equation has imaginary solution (u, v), $v\neq 0$, since the discriminant of the equation is negative. The solution points are obtained by the condition (2.2). In Figure 3. 2, a line $y=(b/2)x+c$ and the x-axis intersect at the point $A(-2c/b, 0)$. Draw a circle S with center $(-c/b, 0)$ and radius c/b. Let M, N be the intersections of S and a line $x=-c/b$. If a moves continuously from $b^2/4c$ to $+\infty$, then the solution points of equation $ax^2+bx+c=0$ move from A to the origin O along the circular arcs AMO, ANO.

Assume the variable a satisfies the condition $0<a<b^2/4c$. Then, the quadratic equation has two non-equal real solutions $(u-v, 0)$, $(u+v, 0)$, where $u:=-b/2a$, $v:=((b/2a)^2-(c/a))^{1/2}$. If a variable a moves continuously from $b^2/4c$ to $+0$, then $u-v\to-\infty$, $u+v\to-(c/b)-0$.

In fact, if a is sufficiently small, by the Maclaurin's expansion we have

$$((b/2a)^2-(c/a))^{1/2}=(b/2a)(1-(4ac/b^2))^{1/2}$$
$$=(b/2a)(1-(2ac/b^2)-(2a^2c^2/b^4)-(4a^3c^3/b^6)-\cdots).$$

Hence, we have

$$u-v=-(b/2a)-((b/2a)^2-(c/a))^{1/2}$$
$$=-(b/a)+(c/b)+(ac^2/b^3)+(2a^2c^3/b^5)+\cdots,$$

from which we see that if $a\to+0$, then $u-v\to-\infty$.

Similarly, we have

$$u+v=-(b/2a)+((b/2a)^2-(c/a))^{1/2}$$
$$=-(c/b)-(ac^2/b^3)-(2a^2c^3/b^5)-\cdots$$
$$<-c/b,$$

from which we see that if $a\to+0$, then $u+v\to-c/b$.

Example 3. 1. Consider a quadratic equation $ax^2+2x+5=0$, $a>0$. Then, $b^2/4c=1/5$, $-2c/b=-5$. Numerical values of the solutions for some a are as follows.

a	Solution point $(u, -v)$	Solution point (u, v)
10000	(−0.0001, −0.0224)	(−0.0001, 0.0224)
1000	(−0.0010, −0.0707)	(−0.0010, 0.0707)
100	(−0.0100, −0.2234)	(−0.0100, 0.2234)
10	(−0.1000, −0.7000)	(−0.1000, 0.7000)
5	(−0.2000, −0.9798)	(−0.2000, 0.9798)
4	(−0.2500, −1.0897)	(−0.2500, 1.0897)
3	(−0.3333, −1.2472)	(−0.3333, 1.2472)
2	(−0.5000, −1.5000)	(−0.5000, 1.5000)
1	(−1.0000, −2.0000)	(−1.0000, 2.0000)
1/2	(−2.0000, −2.4495)	(−2.0000, 2.4495)
2/5	(−2.5000, −2.5000)	(−2.5000, 2.5000)
1/3	(−3.0000, −2.4495)	(−3.0000, 2.4495)
1/4	(−4.0000, −2.0000)	(−4.0000, 2.0000)
1/5	(−5.0000, 0)	(−5.0000, 0)
1/10	(−17.0711, 0)	(−2.9289, 0)
1/100	(−197.4679, 0)	(−2.5321, 0)
1/1000	(−1997.4969, 0)	(−2.5031, 0)
1/10000	(−19997.4997, 0)	(−2.5003, 0)

3. 2. 2. Case of $a>0$, $b>0$, $c=0$

In this case, (2.1) becomes $x(ax+b)=0$, hence, the solutions are $O(0, 0)$, $(-b/a, 0)$. The solution $P(-b/a, 0)$ is obtained as an x-intercept of a line passing through point $(0, b)$, which is parallel to a line joining points $(0, a)$, $(-1, 0)$. We see that

if $a\to+0$, then $P\to(-\infty, 0)$,

if $a\to+\infty$, then $P\to O$.

(Figure 3. 3)

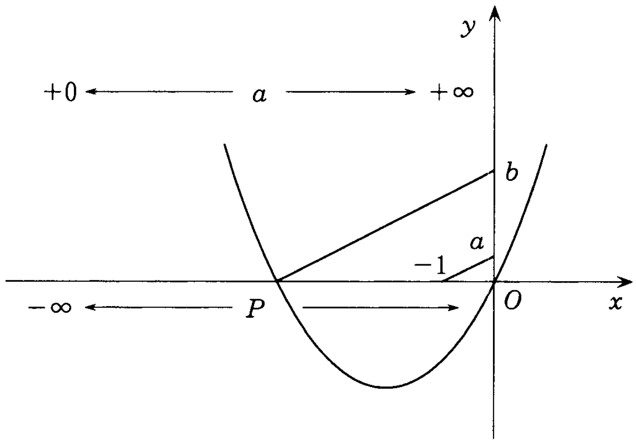

Fig. 3. 3 Variation of solution $P(-b/a, 0)$ for a, where $a>0$, $b>0$, $c=0$

3. 2. 3. Case of $a>0$, $b>0$, $c<0$

The quadratic equation $ax^2+bx+c=0$ has two non-equal real solutions $(u-v, 0), (u+v, 0)$, where $u:=-b/2a$, $v:=((b/2a)^2-(c/a))^{1/2}$, since the discriminant of the equation is positive. Hence, $u-v<0$, $u+v>0$. The inequality $u+v>0$ follows from $(u-v)(u+v)=c/a<0$ thanks to the relation between the solutions and coefficients of quadratic equation.

For sufficiently small a, by the Maclaurin's expansion we have

$u-v=-(b/a)+(c/b)+(ac^2/b^3)+(2a^2c^3/b^5)+\cdots,$

$u+v=-(c/b)-(ac^2/b^3)-(2a^2c^3/b^5)-\cdots,$

therefore, we see that

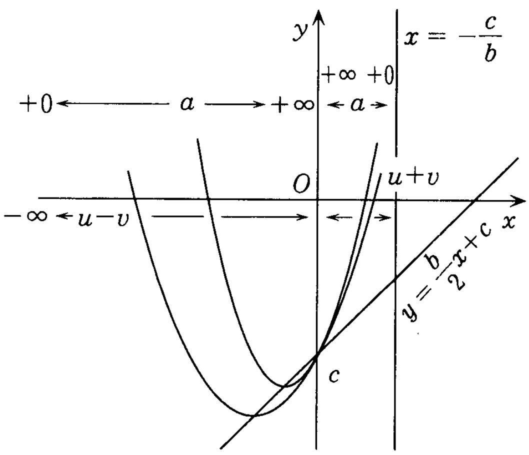

Fig. 3. 4 Variation of solutions for a, where $a>0$, $b>0$, $c<0$

if $a \to +0$, then $u-v \to -\infty$, $u+v \to -c/b$,
and if $a \to +\infty$, then $u-v \to -0$, $u+v \to +0$.
(Figure 3. 4)

Example 3. 2. For a quadratic equation $ax^2+2x-5=0$, $a>0$, numerical values of the real solutions for some a are as follows.

a	Solution point $(u-v, 0)$	$(u+v, 0)$
10000	(−0.0225, 0)	(0.0223, 0)
1000	(−0.0717, 0)	(0.0697, 0)
100	(−0.2338, 0)	(0.2138, 0)
10	(−0.8141, 0)	(0.6141, 0)
1	(−3.4495, 0)	(1.4495, 0)
0.1	(−22.2474, 0)	(2.2474, 0)
0.01	(−202.4695, 0)	(2.4695, 0)
0.001	(−2002.4969, 0)	(2.4969, 0)
0.0001	(−20002.4997, 0)	(2.4997, 0)

4. Variations of solutions of $ax^2+bx+c=0$ for variation of b, while a, c are fixed

For the quadratic equation

$$ax^2+bx+c=0, \qquad (2.1)$$

$a>0$, we consider the variations of solutions for a continuous variation of the non-negative coefficient b, while a, c remain fix. We consider this variation for three cases $c>0$, $c=0$, $c<0$.

4. 1. Case of $a>0$, $b \geqq 0$, $c>0$

In this case, we consider the variation of solutions in the following three subcases:

$0 \leqq b < 2(ac)^{1/2}$, $b=2(ac)^{1/2}$, $b>2(ac)^{1/2}$.

Assume the variable b satisfies the condition $0 \leqq b < 2(ac)^{1/2}$. Then, (2.1) has imaginary solutions (u, v), $v \neq 0$, where $u := -b/2a$, $v := \pm((c/a)-(b/2a)^2)^{1/2}$, since the discriminant of (2.1) is negative. Hence, the solutions are intersections of a line $x=-b/2a$ and a circle $x^2+y^2=c/a$ from § 2.1.2. Let A be an intersection of the circle and the x-axis such that the

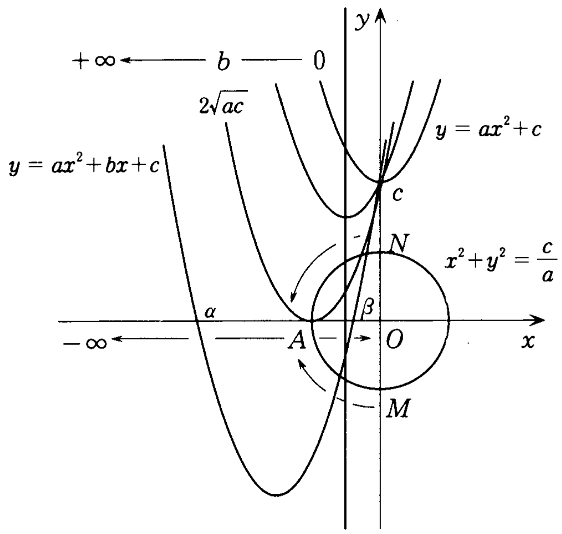

Fig. 4. 1 Variation of solutions for b, where $a>0$, $b\geqq 0$, $c>0$

abscissa of A is negative. Let M, N be the intersections of the circle and the y-axis: $M(0, -(c/a)^{1/2})$, $N(0, (c/a)^{1/2})$. If b moves continuously from 0 to $2(ac)^{1/2}$, then the solution points of equation $ax^2+bx+c=0$ move from M, N to the point A along the circular arcs MA, NA respectively.

Assume $b=2(ac)^{1/2}$, then the solution of (2.1) is a multiple solution $A(-(c/a)^{1/2}, 0)$.

Assume $b>2(ac)^{1/2}$, then the quadratic equation (2.1) has non-equal real solutions $(u-v, 0)$, $(u+v, 0)$, where $u:= -b/2a$, $v:= ((b/2a)^2-(c/a))^{1/2}$.

Then, we see that

if $b\to+\infty$, then $\alpha:=u-v\to-\infty$, $\beta:=u+v\to-0$

by using the relation $\beta=c/a\alpha$, which follows from the relation between solutions and coefficients. (Figure 4. 1)

Example 4. 1. For a quadratic equation $x^2+bx+4=0$, $b \geqq 0$, numerical values of the solutions for some b are as follows.

b	Solution point $(u, -v)$	(u, v)
0	(0, −2.0000)	(0, 2.0000)
1	(−0.5000, −1.9365)	(−0.5000, 1.9365)
2	(−1.0000, −1.7321)	(−1.0000, 1.7321)
3	(−1.5000, −1.3229)	(−1.5000, 1.3229)
4	(−2.0000, 0)	(−2.0000, 0)
5	(−4.0000, 0)	(−1.0000, 0)
6	(−5.2361, 0)	(−0.7639, 0)
7	(−6.3723, 0)	(−0.6277, 0)
8	(−7.4641, 0)	(−0.5359, 0)
9	(−8.5311, 0)	(−0.4689, 0)
10	(−9.5826, 0)	(−0.4174, 0)
100	(−99.9600, 0)	(−0.0400, 0)
1000	(−999.9960, 0)	(−0.0040, 0)
10000	(−9999.9996, 0)	(−0.0004, 0)

4. 2. Case of $a>0$, $b \geqq 0$, $c=0$

The quadratic equation (2. 1) has real solutions $(0, 0), (-b/a, 0)$. Hence, if b moves continuously from 0 to $+\infty$, the point $(-b/a, 0)$ moves from the origin O to $(-\infty, 0)$.

4. 3. Case of $a>0$, $b \geqq 0$, $c<0$

The quadratic equation (2. 1) has non-equal real solutions $(u-v, 0)$, $(u+v, 0)$, where $u:=-b/2a$, $v:=((b/2a)^2-(c/a))^{1/2}$. Clearly, $\alpha:=u-v<0$, hence $\beta:=u+v>0$ since $\beta=c/a\alpha$. Hence, if b moves continuously from 0 to $+\infty$, the solution $(\alpha, 0)$ moves from $(-(-c/a)^{1/2}, 0)$ to $(-\infty, 0)$ and the solution $(\beta, 0)$ moves from $((-c/a)^{1/2}, 0)$ to $(+0, 0)$. (Figure 4. 2)

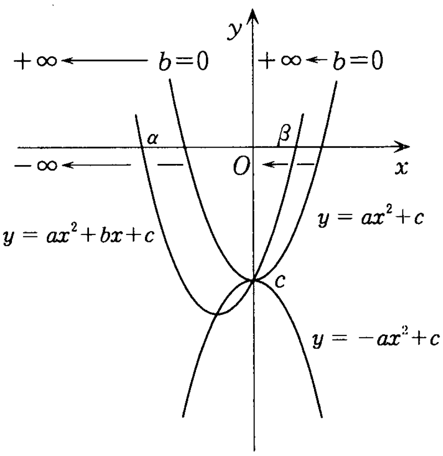

Fig. 4. 2 Variation of solutions for b, where $a>0,\ b\geqq 0,\ c<0$

Example 4. 2. For a quadratic equation $x^2+bx-4=0$, $b\geqq 0$, numerical values of the solutions for some b are as follows.

b	Solution point $(u-v, 0)$	$(u+v, 0)$
0	(−2.0000, 0)	(2.0000, 0)
1	(−2.5616, 0)	(1.5616, 0)
2	(−3.2361, 0)	(1.2361, 0)
3	(−4.0000, 0)	(1.0000, 0)
4	(−4.8284, 0)	(0.8284, 0)
5	(−5.7016, 0)	(0.7016, 0)
6	(−6.6056, 0)	(0.6056, 0)
7	(−7.5311, 0)	(0.5311, 0)
8	(−8.4721, 0)	(0.4721, 0)
9	(−9.4244, 0)	(0.4244, 0)
10	(−10.3852, 0)	(0.3852, 0)
100	(−100.0400, 0)	(0.0400, 0)
1000	(−1000.0040, 0)	(0.0040, 0)
10000	(−10000.0004, 0)	(0.0004, 0)

Remark. We consider the solution of cubic equation $x^3-1=0$ in connection with the locus of imaginary solution of quadratic equation discussed in this section. In [2, § 4.6], we constructed geometrically the solutions of $x^3-1=0$ by construction of imaginary solutions of $x^2+x+1=0$. Here, we consider this problem for $x^2+bx+1=0$ with non-negative parameter b.

Consider a cubic equation $x^3+(b-1)x^2-(b-1)x-1=0$, $b \geqq 0$. This equation can be rewritten as $(x-1)(x^2+bx+1)=0$, hence the solutions are $1, (-b\pm(b^2-4)^{1/2})/2$, that is,

if $0 \leqq b<2$, then solutions are $(1, 0), (-b/2, \pm(1-(b/2)^2)^{1/2})$,

if $b=2$, then solutions are $(1, 0)$, $(-1, 0)$,

if $b>2$, then solutions are $(1, 0)$, $(-(b/2)\pm((b/2)^2-1)^{1/2}, 0)$.

Therefore, if $0 \leqq b<2$, then the solution points are on the circle $x^2+y^2=1$, especially so is the solutions for $b=1$.

5. Variations of solutions of $ax^2+bx+c=0$ for variation of c, while a, b are fixed

For the quadratic equation

$$ax^2+bx+c=0, \tag{2.1}$$

$a>0$, $b \geqq 0$, we consider the variations of solutions for a continuous variation of the coefficient c, while a, b remain fix. We consider these variations for three cases $c>b^2/4a$, $c=b^2/4a$, $c<b^2/4a$.

Assume the variable c satisfies the condition $c>b^2/4a$. Then, the quadratic equation (2.1) has imaginary solutions $(u, -v), (u, v)$, where $u:=-b/2a$, $v:=((c/a)-(b/2a)^2)^{1/2}$. If c moves from $b^2/4a$ to $+\infty$, then the solution points $(u, -v), (u, v)$ move from $(-b/2a, 0)$ to the infinite points $(-b/2a, -\infty)$, $(-b/2a, +\infty)$ along the line $x=-b/2a$, respectively.

Assume $c=b^2/4a$. Then, the quadratic equation (2.1) has a multiple solution $(-b/2a, 0)$, which is a tangent point of the x-axis to parabola $y=ax^2+bx+c$.

Assume c satisfies the condition $c<b^2/4a$. Then, quadratic equation (2.1) has two non-equal real solutions $\alpha:=(u-v, 0)$, $\beta:=(u+v, 0)$, where $u:=-b/2a$, $v:=((b/2a)^2-(c/a))^{1/2}$. If c moves from $b^2/4a$ to $-\infty$, then the solution points α, β move from $(-b/2a, 0)$ to the infinite points $(-\infty, 0)$, $(+\infty, 0)$ along the x-axis, respectively. (Figure 5. 1)

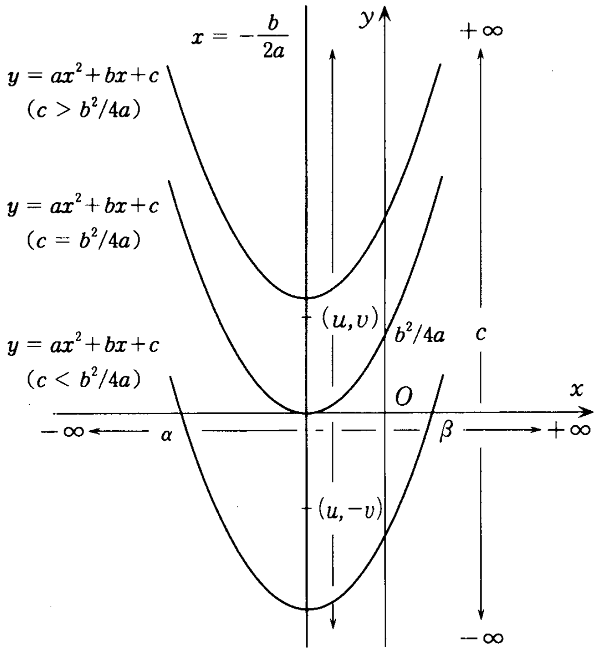

Fig. 5. 1 Variation of solutions for c, where $a>0$, $b\geqq 0$

Example 5. 1. For a quadratic equation $\frac{1}{2}x^2+2x+c=0$, numerical values of the solutions for some c are as follows.

c	Solution point $(u, -v)$	(u, v)
10000	(−2.0000, −141.4072)	(−2.0000, 141.4072)
1000	(−2.0000, −44.6766)	(−2.0000, 44.6766)
100	(−2.0000, −14.0000)	(−2.0000, 14.0000)
10	(−2.0000, −4.0000)	(−2.0000, 4.0000)
5	(−2.0000, −2.4495)	(−2.0000, 2.4495)
4	(−2.0000, −2.0000)	(−2.0000, 2.0000)
3	(−2.0000, −1.4142)	(−2.0000, 1.4142)
2	(−2.0000, 0)	(−2.0000, 0)
1	(−3.4142, 0)	(−0.5858, 0)
0	(−4.0000, 0)	(0.0000, 0)
−1	(−4.4495, 0)	(0.4495, 0)
−2	(−4.8284, 0)	(0.8284, 0)
−3	(−5.1623, 0)	(1.1623, 0)
−4	(−5.4641, 0)	(1.4641, 0)
−5	(−5.7417, 0)	(1.7417, 0)
−10	(−6.8990, 0)	(2.8990, 0)
−100	(−16.2829, 0)	(12.2829, 0)
−1000	(−46.7661, 0)	(42.7661, 0)
−10000	(−143.4355, 0)	(139.4355, 0)

References

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